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SIMPLE SINGULARITIES AND SIMPLE LIE ALGEBRAS*

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Abstract. In this survey article, we summarize the Grothendieck's conjectures relating simple singularities of surfaces and the geometry of finite dimensional complex simple Lie algebras. We also present the research works on the subject up to date with a list of references that can guide a beginner in the subject.

Keywords: simple singularities, resolution of singularities, Lie algebras, subregular nilpotent elements.

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To the memory of V. I. Arnold

1. Introduction

We have tried to be self-contained in these notes, but we have discovered that it is a rather difficult exercise. In fact, roughly, for someone having a basic mathematical culture, these notes are readable. They show that to understand Grothendieck's conjectures relating the geometry of singularities and the geometry of simple Lie algebras, one needs a large understanding of mathematics. Basic knowledge in Commutative algebra can be read in [3] or [20].

In these notes many assertions are quoted as theorems, although they are written nowhere. For instance, the proof of Grothendieck's conjectures is only done in the frame of Lie groups. In [22], P. Slodowy extends some results to simple Lie algebras with a sketch of proof. In these notes we assert that Grothendieck's conjectures extend to simple Lie algebras. We conjecture that this extension is true and we hope that some mathematician will fill up this gap. Below, we also mention Theorem 2.1 without proof. It is an easy consequence of the $\mu*$-theorem of Teissier in [26].

It is well-known (see [21]) that complex simple Lie algebras are of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, i.e. are isomorphic to some matrix algebras. In order to simplify the presentation, we do not give the list of these simple Lie algebras here. We have tried to show that special properties of simple algebras lead to the situation described here and not their specific descriptions.

One has to notice that simple singularities were known as rational surface singularities of embedding dimension 3 by A. Grothendiek (see [5]). V. Arnold understood afterwards the relation with the moduli dimension 0 of these singularities.

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The reader should take these notes as a provocation more than a definitive statement of what can be done in this domain.

The first author contributed from the beginning to the statements of Grothendieck's conjectures. However, later, A. Grothendieck never considered these conjectures to be so deep. We encourage readers to have their own opinion.

2. Simple surface singularities

Let $\mathcal{U}$ be an open neighbourhood of the point 0 in $\mathbb{C}^3$. Let $f : \mathcal{U} \to \mathbb{C}$ be the reduced equation of a surface S, i.e. dim $S = 2$, such that $0 \in S$. It is a consequence of a known result of S. Abhyankar ([1] (45.16) 2) or e.g. [20] Theorem 11, Chap. IV §D) that the dimension 2 hypersurface local ring

$$\mathcal{O}_{S,0} = \mathcal{O}_{\mathbb{C}^3,0}/(f)$$

of S at 0 is normal if and only if the singularity of S at 0 is isolated (Here, for convenience, we consider that the point 0 is an isolated singularity if there is a neighbourhood $\mathcal{U}'$ of 0 in S such that $S \cap \mathcal{U}' \setminus \{0\}$ is non-singular, so an isolated singularity might be non-singular).

We shall consider germs of complex analytic surfaces (S, x) at a point $x \in S$. For convenience, we shall call (S, x) the *singularity* of S at x. In all these notes, we shall mainly consider surfaces which are locally hypersurfaces of $\mathbb{C}^3$.

We say that the surface singularity of $\mathbb{C}^3$ defined at $0 \in \mathcal{V} \subset \mathbb{C}^3$ by a reduced equation $g : \mathcal{V} \to \mathbb{C}$ has the same *topological type* as the surface singularity defined at 0 by the analytic function $f : \mathcal{U} \to \mathbb{C}$, if there is a homeomorphism φ of an open neighbourhood $\mathcal{U}_0$ of 0 in $\mathcal{U}$ onto an open neighbourhood $\mathcal{V}_0$ of 0 in $\mathcal{V}$ such that

$$\varphi(\mathcal{U}_0 \cap \{f = 0\}) = \mathcal{V}_0 \cap \{g = 0\}.$$

We say that the singularity of $\{g = 0\}$ at 0 has the same *analytical type* as the one of $\{f = 0\}$ at 0, if the local rings

$$\mathcal{O}_{\{g=0\},0} \simeq \mathcal{O}_{\{f=0\},0}$$

are isomorphic.

The surface singularity defined at 0 by a reduced equation $f : \mathcal{U} \to \mathbb{C}$, where $\mathcal{U}$ is an open neighbourhood of 0 in $\mathbb{C}^3$, has *no moduli* if any surface singularity which is a hypersurface at the point 0 and which has the same topological type as $\{f = 0\}$ at 0 has the same analytical type as $\{f = 0\}$ at 0.

Lemma 2.1. *Let*

$$\mathcal{A} = \mathbb{C}\{z_1, z_2, z_3\}/(f, \partial f/\partial z_1, \partial f/\partial z_2, \partial f/\partial z_3)$$

be the $\mathbb{C}$-algebra quotient of the local analytic ring $\mathbb{C}\{z_1, z_2, z_3\}$ by the ideal generated by the function f and its partial derivatives $\partial f/\partial z_1$, $\partial f/\partial z_2$ and $\partial f/\partial z_3$. The algebra $\mathcal{A}$ is a finite dimensional complex vector space if and only if the singularity of $\{f = 0\}$ at the origin 0 is isolated.

When the singularity is isolated, we shall denote by τ the dimension of $\mathcal{A}$. The number τ is called the Tjurina number of $\{f = 0\}$ at 0. In the sequel, we shall assume that the surface singularity $\{f = 0\}$ at 0 is isolated.

Let $\sigma_1, \ldots, \sigma_\tau$ be convergent power series in $\mathbb{C}\{z_1, z_2, z_3\}$ whose images in $\mathcal{A}$ give a base of $\mathcal{A}$.

Consider the germ of complex analytic map

$$\Phi : (\mathbb{C}^3 \times \mathbb{C}^\tau, 0) \to (\mathbb{C} \times \mathbb{C}^\tau, 0)$$

defined by

$$\Phi(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) = (f(\mathbf{z}) + \lambda_1 \sigma_1 + \ldots + \lambda_\tau \sigma_\tau, \lambda_1, \ldots, \lambda_\tau).$$

A germ of a map $f : (X, x) \to (S, s)$ is called a deformation of a singularity (Y, y) if $(Y, y) = (f^{-1}(s), x)$ and the local ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module by f (see p. 29 of [3]). Therefore, the germ of map Φ is a deformation of the hypersurface singularity $\{f = 0\}$ at 0, since the germ $(\{f = 0\}, 0)$ is the fiber of Φ over 0 and since the local ring $\mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\tau, 0}$ is regular (see p. 123 of [3]), therefore Cohen-Macaulay (see [20] Chap. IV §B and Chap. IV §D Corollary 3), is a module over the regular local ring $\mathcal{O}_{\mathbb{C} \times \mathbb{C}^\tau, 0}$ by Φ, and the dimension of the fiber of Φ over 0 has the dimension equal to the difference of dimension of the local rings. This last assertion implies that $\mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\tau, 0}$ is flat over $\mathcal{O}_{\mathbb{C} \times \mathbb{C}^\tau, 0}$ by Φ.

In fact, this deformation is the *versal* deformation of the hypersurface singularity $(\{f = 0\}, 0)$ (see [27]). It means that, if $\varphi : (X, x) \to (S, s)$ is a deformation of $(\{f = 0\}, 0)$, where (S, s) is non-singular, there is an analytic map $\sigma : (S, s) \to (\mathbb{C} \times \mathbb{C}^\tau, 0)$ whose tangent map is unique such that φ is the pull-back of Φ by σ.

Hypersurface singularities are usually distinguished by their Milnor number (see [14] Theorem 6.5 or e.g. [12]):

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3, 0} / (\partial f / \partial z_1, \partial f / \partial z_2, \partial f / \partial z_3),$$

the complex dimension of the quotient of $\mathcal{O}_{\mathbb{C}^3, 0}$ by the ideal $(\partial f / \partial z_1, \partial f / \partial z_2, \partial f / \partial z_3)$ generated by the partial derivatives of f.

Note that in general $\tau \leq \mu$. A theorem of K. Saito ([18]) shows that $\tau = \mu$, if and only if after a possible change of analytic coordinates, the reduced equation f of the surface is a quasi-homogeneous polynomial.

For $0 < \eta \ll \varepsilon \ll 1$, we shall call the singularities in the fibers

$$\Phi^{-1}(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \cap B_\varepsilon(0)$$

for $(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \in D_\eta(0)$, the *nearby singularities* of $(\{f = 0\}, 0)$, where $B_\varepsilon(0)$ is the open ball centered at 0 with radius ε in $\mathbb{C}^3 \times \mathbb{C}^\tau$ and $D_\eta(0)$ is the open ball centered at 0 with radius η in $\mathbb{C} \times \mathbb{C}^\tau$.

One can prove the following

Proposition 2.1. *For any $\varepsilon > 0$ small enough, there exists $\eta > 0$ small enough, such that, for $(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \in D_\eta(0)$, the multiplicity of the discriminant of Φ at the point $(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau)$ equals the sum of the Milnor numbers of the singular points of*

$$\Phi^{-1}(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \cap B_\varepsilon(0).$$

Notice that, one can make a natural analytic partition by connected strata of the discriminant of Φ such that the multiplicity of the discriminant is constant on each stratum of the partition.

Even, if one cannot apply the Theorem of [13], one can prove the following theorem by using the $\mu*$-theorem of Teissier in [26]

Theorem 2.1. *For any $\varepsilon > 0$ small enough, there exists $\eta > 0$ small enough, such that the number of topological types of the singularities of the fibers $\Phi^{-1}(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \cap B_\varepsilon(0)$ is finite for $(\mathbf{z}, \lambda_1, \ldots, \lambda_\tau) \in D_\eta(0)$.*

Then we can define

Definition 2.1. *A simple singularity is a surface isolated singularity which has no moduli.*

It has been proved by V.I. Arnold (see e.g. [2], p.205) that

Theorem 2.2. *A simple surface singularity is analytically isomorphic to one of the following singularities at* 0

$$x^2 + y^2 + z^{n+1} = 0 \,(\text{for } n \geq 1,\ A_n \text{ type}),$$
$$x^2 + y^2 z + z^{n-1} = 0 \,(\text{for } n \geq 4,\ D_n \text{ type}),$$
$$x^2 + y^3 + z^4 = 0 \,(E_6 \text{ type}),$$
$$x^2 + y^3 + yx^3 = 0 \,(E_7 \text{ type}),$$
$$x^2 + y^3 + z^5 = 0 \,(E_8 \text{ type}).$$

We distinguish all these singularities by their Milnor number

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,0}/(\partial f/\partial z_1, \partial f/\partial z_2, \partial f/\partial z_3).$$

Notice that the equations of simple surface singularities are quasi-homogeneous.

3. Resolution of surface singularities

We first need the notion of point blowing-up.

For simplicity, we shall only consider point blowing-ups on a curve or on a surface. We consider that our curves or surfaces are reduced. The notion of point blowing-up is general. It is the same for any set defined by a set of complex analytic equations in an open subset of $\mathbb{C}^N$. Such set is called a complex analytic set.

Let $\mathbf{a}$ be a point of a complex analytic set X closed in an open subset of $\mathbb{C}^N$. We have a map

$$\lambda : X \setminus \{\mathbf{a}\} \to \mathbb{P}^{\mathbf{N-1}}$$

defined by $\lambda(x) =$ line $\mathbf{a}$x.

It is easy to see that λ is an analytic map. However this map cannot be extended at $\mathbf{a}$. A natural way to find an extension is to replace the set $\{\mathbf{a}\}$ by the limit set of lines through $\{\mathbf{a}\}$ and a point $x \in X \setminus \{\mathbf{a}\}$. In fact, consider the graph of λ in the product space $X \times \mathbb{P}^{N-1}$. This graph is not a closed subset of the product space.

A theorem of Remmert asserts that the closure of this graph in $X \times \mathbb{P}^{N-1}$ is an analytic space X_1. The projections onto X and $\mathbb{P}^{N-1}$ define complex analytic maps

$$e_1 : X_1 \to X$$
$$\lambda_1 : X_1 \to \mathbb{P}^{N-1}.$$

The map e_1 is called the blowing-up of X at $\mathbf{a}$, the map λ_1 can be considered as an extension of λ since $X \setminus \{\mathbf{a}\}$ embeds isomorphically into X_1 through the graph of λ. Notice that e_1 is an isomorphism of $X_1 \setminus e_1^{-1}(\mathbf{a})$ onto $X \setminus \{\mathbf{a}\}$. The points of $e_1^{-1}(\mathbf{a})$ in $\mathbb{P}^{N-1}$ are the lines of $\mathbb{C}^N$ contained in the tangent cone of X at the point $\mathbf{a}$, since these lines are the limits of the lines line $\mathbf{a}$x when $x \in X \setminus \{\mathbf{a}\}$ tends to $\mathbf{a}$ on X.

When C is a curve, if $C \setminus \{\mathbf{a}\}$ is non-singular, in general C_1 might have many singular points. Their number is bounded by the number of components in the tangent cone of C at 0.

In the case of plane curves, one can show (see e.g. [28] p. 80-81) that after a finite number, say r, of point blowing-ups $e_i : C_i \to C_{i-1}$, one can obtain a non-singular curve C_r.

Let us concentrate on the case of normal surfaces.

In this case, singular points are isolated. Consider a singular point $\mathbf{a}$. Let $e_1 : S_1 \to S$ be the blowing-up of $\mathbf{a}$ in S. In general S_1 is not normal, so the singularities of S_1 might not be isolated. A natural idea is to normalize S_1. More generally if S is not a normal surface, there exists a unique morphism $\nu : \overline{S} \to S$, up to analytic isomorphism, from a normal surface $\overline{S}$ into

S, such that ν is surjective and finite and by ν the space $\overline{S} \setminus \nu^{-1}(N)$ maps isomorphically onto $S \setminus N$, where N is the set of non-normal points of S (see e.g. [4] p. 25).

There is a theorem of R. Walker and O. Zariski (see [29]):

Theorem 3.1. *Let $e_i : S_i \to \overline{S}_{i-1}$ be the blowing-up of the singular points of $\overline{S}_{i-1}$ and $n_i : \overline{S}_i \to S_i$ the normalization of S_i. Suppose that $S_0 := S$, then, there is $r \geq 0$ such that $\overline{S}_r$ is a non-singular surface.*

In some cases the surface S_1 is normal.

Proposition 3.1. *For simple surface singularities, the blowing-up of a simple singularity is a normal surface which is locally a hypersurface and the singularities of the blowing-up surface are also simple surface singularities.*

This proposition and the Theorem of R. Walker and O. Zariski show that a simple singularity is desingularized after a finite number of point blowing-ups, because the sum of Milnor numbers of the singularities decreases strictly after a point blowing-up.

Let $e_1 : S_1 \to S$ be the blowing-up of the point $\mathbf{a}$ in a normal surface S. Then, as we have mentioned before

Proposition 3.2. *The set $e_1^{-1}(\{\mathbf{a}\})$ is the projectivization of the tangent cone of S at $\mathbf{a}$.*

This proposition gives us an easy way to find the exceptional fiber $e_1^{-1}(\{\mathbf{a}\})$ in the case of a surface which is a complex hypersurface in $\mathbb{C}^3$. Suppose $\mathbf{a} = (a_1, a_2, a_3)$. Let

$$f = f_m + f_{m+1} + \dots$$

be the expansion of f in homogeneous polynomials in $z_1 - a_1, z_2 - a_2, z_3 - a_3$. The tangent cone at $\mathbf{a}$ is given by $f_m(z_1, z_2, z_3) = 0$. It defines a projective curve Γ in $\mathbb{P}^2$. Then, as sets $e_1^{-1}(\{\mathbf{a}\}) = \{\mathbf{a}\} \times \Gamma$.

In the case of a simple singularity the algebraic set given by $e_1^{-1}(\{\mathbf{a}\})$ is isomorphic to a projective line, except for A_n simple singularities, where it is isomorphic to two transverse projective lines.

A complex analytic map $\pi : \tilde{S} \to S$ is the *desingularization* or *resolution* of a surface isolated singularity $0 \in S$ if

(1) the space $\tilde{S}$ is non-singular,
(2) it is a proper map,
(3) it induces a complex analytic isomorphism of the complement $\tilde{S} \setminus \pi^{-1}(0)$ onto $S \setminus \{0\}$.

Another way to formulate R. Walker and O. Zariski result quoted above is

Proposition 3.3. *One obtains a desingularization of a surface isolated singularity with the composition of a finite number of normalizations of point blowing-ups.*

If π is a resolution of a singularity $0 \in S$, the fiber $\pi^{-1}(0)$ is called the *exceptional curve* and the components of $\pi^{-1}(0)$ are called the *exceptional components* of π.

Let $\pi : \tilde{S} \to S$ be the desingularization of a simple singularity $0 \in S$. One can prove that $\pi^{-1}(0)$ is a union of non-singular rational curves.

Each of these curves, isomorphic to a projective line, is embedded in $\tilde{S}$. Their embeddings is characterized by their normal bundles in $\tilde{S}$.

We define the *self-intersection* of a component D of $\pi^{-1}(0)$ (which a non-singular rational curve) to be

$$D.D = deg_D(\mathcal{N}_{D|\tilde{S}}),$$

where $\mathcal{N}_{D|\tilde{S}}$ is the normal bundle of D in $\tilde{S}$, or equivalently $D.D = deg_D(\mathcal{L}(D) \otimes_{\mathcal{O}_{\tilde{S}}} \mathcal{O}_D)$ (see e.g. exercize 6.12 p.149 in [9] or p. 67 of [4]), where $\mathcal{L}(D)$ is the sheaf of meromorphic functions on $\tilde{S}$ having a pole along D. A theorem of Du Val shows that the self-intersection of an exceptional component is negative.

It is convenient to consider the dual graph of the resolution π of a simple singularity $0 \in S$, i.e. the weighted graph whose vertices are the components of $\pi^{-1}(0)$ and the number of edges between two vertices equals the intersection number of the components corresponding to the vertices and the weight of a vertex is the self-intersection of the components corresponding to this vertex.

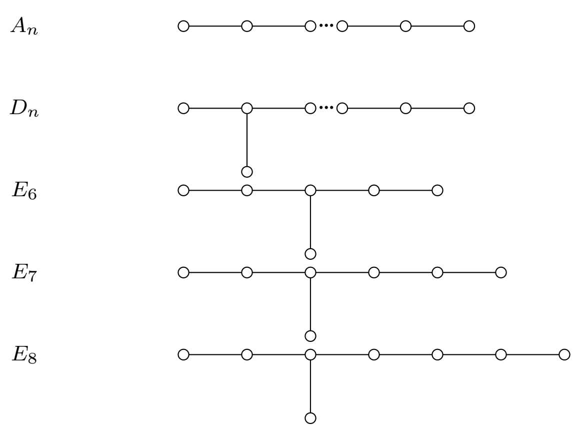
By a theorem of Castelnuovo, one can obtain a resolution of a simple singularity where none of the exceptional curves have self-intersection -1. So, we may assume that in the resolution π the exceptional components have self-intersection ≤ -2. Such a desingularization is called a *minimal* resolution of the simple singularity.

One can prove that $(S, 0)$ is a simple singularity if and only if there is a resolution where all the exceptional components are non-singular rational curves with self-intersection -2 and the dual graph of the resolution is a tree.

This is consequence of a classification Theorem of Coxeter, because we know that the intersection matrix of the exceptional components is definite negative (Du Val Theorem, see e.g.[15]). In fact the Theorem of Coxeter gives all the quadratic forms which are definite positive and whose symmetric matrix has integer entries equal to 2 in the diagonal and 0 or -1 outside (see e.g. [6] p. 459)

This point is important to recognize simple singularities in simple Lie algebras.

In fact, let $\pi : \tilde{S} \to S$ be a minimal desingularization of a simple surface singularity. One can prove that the dual graphs (without weights) of their minimal resolution are the Dynkin diagrams (without weights) of the simple Lie algebras A_n, D_n or E_n, in the case of the simple singularities A_n, D_n or E_n given above in Theorem 2.2.



This indicates a relation between simple singularities and simple Lie algebras. We shall explain this relation below.

4. Lie algebras

Most of the concepts described here can be found in [21].

A complex Lie algebra $\mathcal{L}$ is a complex vector space with a multiplication $[\,,]; \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which is bilinear for the complex vector space structure and which satisfies the following axioms:

$[x, x] = 0$ for all $x \in \mathcal{L}$,

$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathcal{L}$ (Jacobi identity).

As a first consequence, Lie algebras are anti-commutative.

Definition 4.1. *A Lie subalgebra $\mathcal{L}_1$ of a Lie algebra $\mathcal{L}$ is a complex vector subspace such that, for all $x, y \in \mathcal{L}_1$, we have $[x, y] \in \mathcal{L}_1$.*

Notation: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lie subalgebras of $\mathcal{L}$. We shall denote $[\mathcal{L}_1, \mathcal{L}_2]$ the Lie algebra generated by the products $[b_1, b_2]$, for all $b_1 \in \mathcal{L}_1$ and for all $b_2 \in \mathcal{L}_2$.

Definition 4.2. *An ideal $\mathcal{I}$ of a Lie algebra $\mathcal{L}$ is a complex Lie subalgebra such that $[\mathcal{I}, \mathcal{L}] \subset \mathcal{I}$.*

Since the first axiom above implies $[x, y] = -[y, x]$, we do not need to distinguish between left and right ideals and $[\mathcal{I}, \mathcal{L}] = [\mathcal{L}, \mathcal{I}]$.

If $\mathcal{L}$ is a finite dimensional Lie algebra and $\mathcal{I}$ is an ideal of $\mathcal{L}$, then the quotient $\mathcal{L}/\mathcal{I}$ is a Lie algebra of complex dimension $\dim \mathcal{L} - \dim \mathcal{I}$.

A map $\rho : \mathcal{L}_1 \to \mathcal{L}_2$ of complex Lie algebras is a morphism of complex Lie algebras if it is a complex linear homomorphism and, for any $x, y \in \mathcal{L}_1$, $\rho([x, y]) = [\rho(x), \rho(y)]$.

Examples.

1. Lie Algebra of vector fields on a manifold (in general infinite dimensional).
2. Lie Algebra of derivations of a ring of smooth functions (idem).
3. Endomorphisms $End(V)$ of a vector space V have a natural Lie algebra structure with

$$[f, g] = f \circ g - g \circ f$$

(finite dimensional if V is finite dimensional).

We say that $\mathcal{L}$ is an *abelian* Lie algebra if $[\mathcal{L}, \mathcal{L}] = 0$.

Consider the sequence $\mathcal{L}_1 = \mathcal{L}, \ldots, \mathcal{L}_{i+1} := [\mathcal{L}_i, \mathcal{L}], \ldots$. We say that $\mathcal{L}$ is *nilpotent* if there is $r > 0$ such that $\mathcal{L}_{r+1} = 0$.

Consider now the following sequence: $\mathcal{L}_1 = \mathcal{L}, \ldots, \mathcal{L}_{i+1} := [\mathcal{L}_i, \mathcal{L}_i], \ldots$. We say that $\mathcal{L}$ is *solvable* if there is $r > 0$ such that $\mathcal{L}_{r+1} = 0$.

Example. Let V be a finite dimensional complex vector space. Consider the Lie algebra of the endomorphisms of V. The subalgebra $\mathfrak{E}(\mathbf{D})$ of endomorphisms which leaves invariant a flag $\mathbf{D}$ of V is solvable.

Lemma 4.1. *Let $\mathcal{L}$ be a Lie algebra. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two solvable ideals of $\mathcal{L}$. Then $[\mathcal{I}_1, \mathcal{I}_2]$ is a solvable ideal of $\mathcal{L}$.*

Then, a Lie algebra $\mathcal{L}$ has a unique maximal solvable ideal $\mathcal{R}$ called the *radical* of $\mathcal{L}$.

Definition 4.3. *The Lie algebra $\mathcal{L}$ is semi-simple if its radical ideal is $\{0\}$.*

There is an important characterization of semi-simple Lie algebras using the Killing form.

First notice that a Lie algebra $\mathcal{L}$ has a representation in itself by the homomorphism

$$ad : \mathcal{L} \to End(\mathcal{L})$$

defined by $ad(x)(y) = [x, y]$ for any $x, y \in \mathcal{L}$ and where $End(\mathcal{L})$ is endowed with the natural Lie algebra structure mentioned above.

Definition 4.4. *The Killing form of the Lie algebra $\mathcal{L}$ is the bilinear form of $\mathcal{L}$ defined by*

$$< x, y >:= Tr(ad(x) \circ ad(y)).$$

The Killing form is symmetric and associative

$$\forall x, y, z \in \mathcal{L}, < [x, y], z >=< x, [y, z] > .$$

Theorem 4.1. *The complex Lie algebra $\mathcal{L}$ is semi-simple if and only if its Killing form is non-degenerate.*

Now, we can define

Definition 4.5. *A Lie algebra $\mathcal{L}$ is said to be simple if its only ideals are $\{0\}$ and $\mathcal{L}$.*

Since the orthogonal of an ideal $\mathcal{I}$ by a bilinear form which is symmetric and invariant, is an ideal $\mathcal{I}^{\perp}$ and since the Killing form of a semi-simple Lie algebra is non-degenerate, one can prove that a semi-simple Lie algebra is isomorphic to the product of its minimal ideals, which are obviously simple Lie algebras. We have:

If $\mathcal{L}$ is a direct sum of simple ideals $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$ as Lie algebras, then every simple ideal of $\mathcal{L}$ coincides with one of $\mathcal{L}_i$'s. In that case, $\mathcal{L}$ is semisimple.

Proposition 4.1. *A complex Lie algebra $\mathcal{L}$ is semi-simple if and only if there are simple ideals $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_r$.*

In fact we have the following more general result.

Proposition 4.2. *Any ideal and any quotient of a semi-simple Lie algebra is semi-simple.*

A nice property of complex semi-simple Lie algebras is given by the following theorem of H. Weyl:

Let $\mathcal{L}$ be a complex semi-simple Lie algebra. An element x of $\mathcal{L}$ is said to be *semi-simple* if $ad(x)$ is a semi-simple endomorphism of $\mathcal{L}$ (i.e. diagonalizable over $\mathbb{C}$) and said to be *nilpotent* if $ad(x)$ is a nilpotent endomorphism of $\mathcal{L}$.

Theorem 4.2. *Let $\mathcal{L}$ be a semi-simple Lie algebra. Every element x of $\mathcal{L}$ is uniquely written $s+n$ where s is semi-simple and n is nilpotent and $[s, n] = 0$ Any element which commutes with x commutes with s and n.*

Moreover, let $\rho : \mathcal{L} \to End(V)$ be a representation of $\mathcal{L}$, i.e. a homomorphism of Lie algebras, if x is nilpotent (resp. semi-simple), $\rho(x)$ is nilpotent (resp. semi-simple).

Then,

Theorem 4.3. *Every finite dimensional representation of a semi-simple Lie algebra is completely reducible.*

5. Classification of simple Lie algebras

Here we are going to give a classification of complex simple algebras (see e.g. [21] or [10]).

The normalizer of a subalgebra $\mathcal{L}'$ of $\mathcal{L}$ is the biggest subalgebra of $\mathcal{L}$ which contains $\mathcal{L}'$ as an ideal, i.e. $\{x \in \mathcal{L}, [x, \mathcal{L}'] \subset \mathcal{L}'\}$.

Definition 5.1. *A subalgebra of a Lie algebra $\mathcal{L}$ is called Cartan subalgebra if it is nilpotent and equals its normalizer in $\mathcal{L}$.*

Then,

Theorem 5.1. *Any finite dimensional Lie algebra $\mathcal{L}$ has a Cartan subalgebra.*

Let $x \in \mathcal{L}$. Consider the Lie subalgebra of $\mathcal{L}$:

$$\mathcal{L}_0(x) := \{y \in \mathcal{L} \mid \exists k, (ad(x))^k(y) = 0\}.$$

Notice that $x \in \mathcal{L}_0(x)$.

Definition 5.2. *The element x is regular if the dimension $\dim \mathcal{L}_0(x)$ is the smallest possible.*

Then, one can prove:

Lemma 5.1. *$\mathcal{L}_0(x)$ is a Cartan subalgebra if and only if x is regular.*

Observations. If the Lie algebra $\mathcal{L}$ is nilpotent, then $\mathcal{L}$ is a Cartan subalgebra of itself.

Any two Cartan subalgebras of $\mathcal{L}$ are conjugate.

Any Cartan subalgebra of $\mathcal{L}$ equals $\mathcal{L}_0(x)$ for some regular element x.

One can define the rank of a semi-simple Lie algebra as the dimension of a Cartan subalgebra.

We also need some important facts about representations of nilpotent Lie algebras: Let $\mathcal{N}$ be a nilpotent Lie algebra. Let $\mathfrak{M}$ be a finite dimensional complex vector space which is a $\mathcal{N}$-module, that we shall call a finite dimensional $\mathcal{N}$-module.

Definition 5.3. *Let $\mathcal{N}$ be a nilpotent Lie algebra. A weight of a finite dimensional $\mathcal{N}$-module $\mathfrak{M}$ is a linear map $\lambda : \mathcal{N} \to \mathbb{C}$ for which there is $m \in \mathfrak{M}$, $m \neq 0$, such that*

$$\forall n \in \mathcal{N}, \exists r > 0, (ad(n) - \lambda(n))^r(m) = 0.$$

Proposition 5.1. *Let $\mathcal{N}$ be a nilpotent Lie algebra. A finite dimensional $\mathcal{N}$-module $\mathfrak{M}$ has a finite number of weights which all vanish on $[\mathcal{N}, \mathcal{N}]$.*

Let $\mathcal{N}$ be a nilpotent Lie algebra. Let α be a weight of a finite dimensional $\mathcal{N}$-module $\mathfrak{M}$. Define $\mathfrak{M}_\alpha$ to be the submodule

$$\{m \in \mathfrak{M} \mid \forall n \in \mathcal{N}, \exists r > 0, (ad(n) - \alpha(n))^r(m) = 0\}.$$

Then, $\mathfrak{M} = \oplus_\alpha \mathfrak{M}_\alpha$.

Let $\mathfrak{H}$ be a Cartan subalgebra of the finite dimensional Lie algebra $\mathcal{L}$. Since $[\mathfrak{H}, \mathcal{L}] \subset \mathcal{L}$, the Lie algebra $\mathcal{L}$ is a $\mathfrak{H}$-module. The Cartan subalgebra $\mathfrak{H}$ being nilpotent, if A is the set of weights of the $\mathfrak{H}$ -module $\mathcal{L}$, we have

$$\mathcal{L} = \oplus_{\alpha \in A} \mathcal{L}_\alpha.$$

Definition 5.4. *We call the set $A \setminus \{0\}$ the set of roots R of $\mathcal{L}$ and, for $\alpha \in R$ the space $\mathcal{L}_\alpha$ is the corresponding root space.*

Consider the case $\mathcal{L}$ is semi-simple and finite dimensional over the field of complex numbers. Then

(1) $\mathcal{L}_0 = \mathfrak{H}$;
(2) $[\mathfrak{H}, \mathfrak{H}] = 0$, i.e. $\mathfrak{H}$ is abelian;
(3) if α is a root, then $-\alpha$ is a root and if $k\alpha$ is a root, then, $k = +1$ or -1;
(4) assume that α and β are linearly independent roots, if $\alpha+\beta$ is not a root, then $[\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0$. If $\alpha + \beta$ is a root, then $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta}$. Also $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subset \mathfrak{H}$;
(5) every element of $\mathfrak{H}$ is semi-simple;
(6) the restriction to $\mathfrak{H}$ of the Killing form is non-degenerate;
(7) $\mathcal{L} = \mathfrak{H} \oplus_{r \in R} \mathcal{L}_r$ and $\dim \mathcal{L}_r = 1$.

Now let us choose a Cartan subalgebra $\mathfrak{H}$ of the simple Lie algebra $\mathcal{L}$. Let $\mathfrak{R}$ be the root system defined by $\mathfrak{H}$. Choose a basis among the roots, say $r_1, \ldots r_\ell$, where ℓ is the rank of $\mathcal{L}$. Any other root is a rational linear combination of this basis. Let us denote by $\mathfrak{H}_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space that they generate in $\mathfrak{H}$.

The Killing form defines a natural isomorphism between $\mathfrak{H}$ and its dual $\mathfrak{H}^*$ such that

$$r(h) =< r^*, h > .$$

Let us identify $\mathfrak{H}$ and its dual $\mathfrak{H}^*$ by this correspondance and write r instead of r^*. It can be shown by this identification that the roots generate $\mathfrak{H}*$.

One can show that the restriction of the Killing form to $\mathfrak{H}_{\mathbb{Q}}$ is definite positive.

Now, let us define an ordering in $\mathfrak{H}_{\mathbb{Q}}$, e.g. $a > b$ if $a - b = \sum_{i=1}^{\ell} \lambda_i r_i$, the first $\lambda_i \neq 0$ is > 0.

By defining an ordering on $\mathfrak{H}_{\mathbb{Q}}$, we define the positive roots R_+ and the negative roots R_-. Choose the fundamental roots to be positive roots such that they are not sum of two positive roots. The set of fundamental roots is also a basis $p_1 \ldots, p_\ell$ of $\mathfrak{H}_{\mathbb{Q}}$.

We denote

$$\mathfrak{B} = \mathfrak{H} \oplus_{r \in R_+} \mathcal{L}_r,$$

$$\mathfrak{N} = \oplus_{r \in R_+} \mathcal{L}_r,$$

$$\mathfrak{N}_- = \oplus_{r \in R_-} \mathcal{L}_r.$$

Lemma 5.2. (1) *$\mathfrak{N}$ and $\mathfrak{N}_-$ are nilpotent subalgebras of $\mathcal{L}$;*
(2) *$\mathfrak{B}$ is a solvable subalgebra of $\mathcal{L}$.*

Definition 5.5. *The Lie algebra $\mathfrak{B}$ is called the Borel subalgebra of $\mathcal{L}$ associated to the Cartan subalgebra $\mathfrak{H}$ and and the root system R.*

For a finite dimensional simple Lie algebra $\mathcal{L}$, we have the decomposition $\mathfrak{B} = \mathfrak{H} \oplus \mathfrak{N}$, where $\mathfrak{N}$ is a nilpotent subalgebra of $\mathcal{L}$, which is called *Levi decompostion.* One can show:

Theorem 5.2. *Let $\mathcal{L}$ be a finite dimensional complex simple Lie algebra. The root systems R_1 and R_2 associated with two Cartan subalgebras $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of $\mathcal{L}$ respectively are isomorphic.*

This theorem shows that, given a simple Lie algebra $\mathcal{L}$, we may consider any root system of $\mathcal{L}$.

Lemma 5.3. *For distinct fundamental roots, we have $< p_i, p_j > \leq 0$.*

We associate to each root a hyperplane H_r of $\mathfrak{H}_{\mathbb{Q}}$ orthogonal to r for the Killing form. We have the symmetry of $\mathfrak{H}_{\mathbb{Q}}$ associated to this hyperplane

$$w_r : \mathfrak{H}_{\mathbb{Q}} \to \mathfrak{H}_{\mathbb{Q}},$$

given by

$$w_r(x) = x - \frac{2 < r, x >}{< r, r >} r.$$

It can be shown that, $\forall r, s \in R$, $w_r(s) \in R$.

Definition 5.6. *The group W generated in the group of linear automorphism of $\mathfrak{H}$ by the w_r, for $r \in R$, is called the Weyl group of $\mathcal{L}$.*

It can be shown that W is generated by the fundamental reflections $w_{p_1}, \ldots, w_{p_\ell}$ but is not generated by any proper subset of the set of these fundamental reflections.

Lemma 5.4. *The numbers*

$$A_{i,j} := \frac{2 < p_i, p_j >}{< p_i, p_i >}$$

are integers.

These numbers $A_{i,j}$ are called the *Cartan numbers.* The *Cartan matrix* is the matrix whose entries are the $A_{i,j}$'s.

Corollary 5.1. *The cosine of the angle $\theta_{i,j}$ between p_i and p_j satisfies*

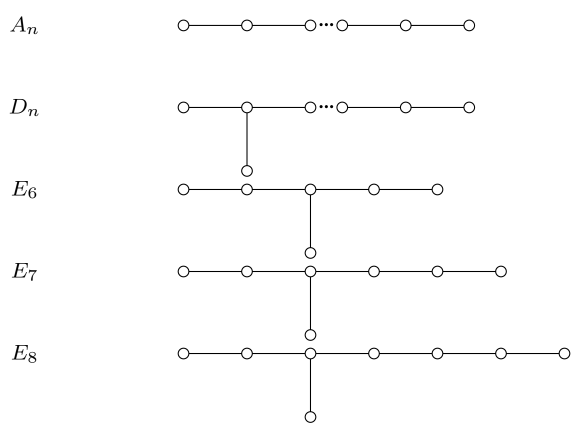
$$\cos^2(\theta_{i,j}) = \frac{A_{i,j}}{2}\frac{A_{j,i}}{2}$$

so if $i \neq j$, $\cos^2(\theta_{i,j}) = 0, 1/4, 1/2, 3/4$.

The Dynkin diagram without weight of the finite dimensional simple Lie algebra $\mathcal{L}$ is the graph given by vertices associated to each fundamental root p_i and with $4\cos^2(\theta_{i,j})$ edges between the vertices (p_i) and (p_j).

In this way one obtains the graphs A_n, $B_n = C_n$, D_n, E_6, E_7, E_8, , F_4, G_2 which are the Dynkin diagrams without weight of the simple Lie algebra $\mathcal{L}$ of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 (see [10] §5 and 6 of Chapter IV).

However, when the Cartan matrix is symmetric, the theorem of Coxeter, quoted above at the end of the section 2, shows that we only have the Dynkin diagrams (without weight) which correspond to the simple Lie algebras of type A_n, D_n, E_6, E_7, and E_8, in which cases $4\cos^2(\theta_{i,j}) = 0, 1$:



These are precisely the same diagrams which are the dual graphs of the exceptional fiber of the minimal resolution of simple singularities.

The relation between simple singularities and simple Lie algebras is explained by Grothendieck's conjectures.

6. Grothendieck's conjectures

Consider a simple complex Lie algebra $\mathcal{G}$ of type A_n, D_n, E_6, E_7 or E_8 and G the corresponding simply connected simple Lie group. Let us fix a Cartan subalgebra $\mathfrak{H}$ of $\mathcal{G}$ and $\mathfrak{B}$ a Borel subalgebra of $\mathcal{G}$. Let B the corresponding Borel subgroup of G whose Lie algebra is $\mathfrak{B}$.

One has a natural map γ of $\mathcal{G}$ into $\mathfrak{H}/W$, where W is the Weyl group, defined, for $x \in \mathcal{G}$, by:

$$\gamma(x) = [\text{conjugate of the semi-simple part of } x \text{ in } \mathfrak{H} \text{ modulo } W].$$

$$\begin{array}{rcl}\gamma : \mathcal{G} & \longrightarrow & \mathfrak{H}/W \\ x & \mapsto & (\gamma_1(x), \dots, \gamma_r(x)),\end{array}$$

where $\gamma_1, \dots, \gamma_r$ are the homogeneous G-invariant polynomials generating $\mathbb{C}[\mathcal{G}]^G$. The map γ is called the *adjoint quotient map*.

Each fiber of γ consists of finitely many orbits, has codimension r in $\mathcal{G}$ and contains an orbit which is dense in the fibre and which coincide with the non-singular points of the fiber (see [11] or e.g. [22], p.31 for Lie groups).

Definition 6.1. *The fiber $\gamma^{-1}(\gamma(0))$ is called the nilpotent variety of $\mathcal{G}$.*

We denote by $\mathcal{N}$ the nilpotent variety of $\mathcal{G}$. By [11] Theorem 0.8, $\mathcal{N}$ has normal singularities.

The first part of Grothendieck's conjectures consists in stating that the morphism γ has a simultaneous resolution (see [22] §4).

Let us remind what is a simultaneous resolution of a morphism.

Definition 6.2. *A simultaneous resolution of a morphism of reduced algebraic varieties $\chi : X \to S$ is given by a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \chi} \\ S' & \xrightarrow{\psi} & S \end{array}$$

such that

(1) *θ is smooth, i.e. flat with non-singular fibers;*
(2) *ψ is finite and surjective;*
(3) *φ is proper;*
(4) *for all $t \in S'$, the morphism $\varphi_t : \theta^{-1}(t) \to X_{\psi(t)}$ induced by φ is a resolution of singularities of the reduced fiber $X_{\psi(t)} := \chi^{-1}(\psi(t))$.*

As above for surfaces, if X is an algebraic variety, an algebraic morphism $\pi : \tilde{X} \to X$ is a resolution of singularities, if (1) $\tilde{X}$ is non-singular, (2) π is proper, (3) π induces an isomorphism of $\tilde{X} \setminus \pi^{-1}(\Sigma)$ onto $X \setminus \Sigma$, where Σ is the subset of singular points of X.

Let H be a closed subgroup of the simple Lie group G. We have the principal fiber bundle $G \to G/H$ over the base G/H with structure group H. Let X be a reduced variety on which H operates regularly. We can define, for $G \to G/H$, the associated fiber bundle $G \times^H X$ over G/H with fiber X as the quotient of $G \times X$ by the H-action

$$H \times G \times X \to G \times X,$$

given by $(h, g, x) \mapsto (gh^{-1}, hx)$ (see e.g. [22] 3.7).

Now, suppose that H is a Borel subgroup B of G. Since B operates on its Lie algebra $\mathfrak{B}$ by the adjoint map, we can define the non-singular space $G \times^B \mathfrak{B}$. Since $\mathfrak{B}$ is a Lie subalgebra of $\mathcal{G}$ and the inclusion $\mathfrak{B} \subset \mathcal{G}$ is B-equivariant, there is a G-equivariant inclusion $G \times^B \mathfrak{B} \hookrightarrow G \times^B \mathcal{G}$ (see §3.7 of [22]). Lemma 1 of §3.7 of [22] shows that, since the B-action on $\mathcal{G}$ is the restriction of the adjoint action which is a G-action, by considering the diagonal G-action on $G/B \times \mathcal{G}$. We obtain that $G \times^B \mathcal{G}$ is G-equivariant isomorphic to $G/B \times \mathcal{G}$. Therefore $G \times^B \mathfrak{B}$ embeds in $G/B \times \mathcal{G}$.

In [22] §4, P.Slodowy proves that there is a natural smooth map $\tilde{\gamma} : G \times^B \mathfrak{B} \to \mathfrak{H}$. The preceding argument gives, by the second projection of $G/B \times \mathcal{G}$ onto $\mathcal{G}$, a map:

$$G \times^B \mathfrak{B} \xrightarrow{\phi} \mathcal{G}$$

Then P. Slodowy shows that the following diagram is commutative

$$\begin{tikzcd} G \times^B \mathfrak{B} \arrow[r, "\phi"] \arrow[d, "\tilde{\gamma}"] & \mathcal{G} \arrow[d, "\gamma"] \\ \mathfrak{H} \arrow[r] & \mathfrak{H}/W \end{tikzcd}$$

The following result conjectured by A. Grothendieck [8] is proved by P. Slodowy in [22].

Theorem 6.1. *A simultaneous resolution for γ is given by the preceding diagram.*

In [23] (Proposition 2.1 and 2.5) T. Springer gave a resolution of the nilpotent variety $\mathcal{N} := \gamma^{-1}(0)$ of $\mathcal{G}$:

$$G \times^B \mathfrak{N} \xrightarrow{\pi} \mathcal{N},$$

where $\mathfrak{N}$ is the subalgebra defined in Lemma 5.2. The map ϕ given above is a generalization of Springer resolution, since it gives a resolution of the fibers of γ.

If $y \in \mathcal{N}$, by definition the conjugate of the semisimple part of y in $\mathfrak{H}$ modulo the Weyl group W is 0, so the semisimple part of y is 0. Let Ad be the adjoint representation of G, i.e. the map $G \to Aut(\mathcal{G})$, $g \mapsto Ad(g)$, where $Ad(g)$ is the derivative at the unit element of G of the conjugation in G by g. Let $x \in G$. The semisimple part of $Ad(x)(y)$ is also 0. The algebraic group G operates on the variety $\mathcal{N}$ by $G \times \mathcal{N} \to \mathcal{N}$, where $(x, y) \mapsto Ad(x)(y)$. The variety $\mathcal{N}$ is a finite union of orbits of the action of G.

One of its orbits is dense in $\mathcal{N}$, namely the orbit composed of the non-singular points of $\mathcal{N}$ as proved by R. Steinberg in [24]. The elements of this orbit are called *regular* elements of $\mathcal{N}$. In the singular subset of $\mathcal{N}$, R. Steinberg showed that there is a codimension 2 orbit in $\mathcal{N}$ which is composed of singular points of $\mathcal{N}$ called *subregular* elements (see 3.10 of [25]). Let y be a subregular element. Let X be a non-singular slice of the subregular element orbit O, i.e. a nonsingular complex analytic submanifold of the algebra $\mathcal{N}$ of codimension dim O transverse in $\mathcal{G}$ to O at a subregular element.

Remark 6.1. *Let $x \in \mathcal{G}$. The dimension of the orbit $G.x$ is even* (see e.g. [11] Proposition 0.5).

Here we are interested in the simple singularities. So we consider a Lie algebra $\mathcal{G}$ of type A_n, D_n, E_6, E_7 or E_8.

Grothendieck [8] (see also [5]) also conjectured that:

Theorem 6.2. *With the above notation, let the simple Lie algebra $\mathcal{G}$ be of type A_n, D_n, E_6, E_7 or E_8. Let y be a subregular nilpotent element and X a transversal slice at y to the orbit $G.y$. The germ of surface $X \cap \mathcal{N}$ at y is a simple surface singularity whose type is A_n, D_n or E_n for the Lie algebra of the same type.*

Furthermore, in these cases the map γ induces a germ of map $(X, y) \to (\mathfrak{H}/W, 0)$ which is the versal deformation of the surface singularity $(X \cap \mathcal{N}, y)$.

A sketch of proof was given by E. Brieskorn in [5]. A complete, but different, proof was given by H. Esnault in [7].

A consequence of the Theorem 6.1 above is that, if y is a nilpotent subregular element of $\mathcal{N}$, the fiber $\phi^{-1}(y)$ is the exceptional curve of a resolution of the surface singularity $(X \cap \mathcal{N}, y)$. H. Esnault proved that this resolution is minimal and the dual graph of the exceptional curve $\phi^{-1}(y)$ is precisely the Dynkin diagram of the simple Lie algebra $\mathcal{G}$.

Other results concern non-isolated singularities. One may consider some other irregular point x of $\mathcal{N}$ and a slice $\mathfrak{S}$ transverse to the orbit $G.x$ at x. The singularity $\mathfrak{S} \cap \mathcal{G}$ at x has dimension $2k$ where $r + 2k$ is the codimension of the orbit $G.x$ in $\mathcal{G}$ and r is the rank of the group G, i.e. $\dim \mathfrak{H}$. In general these singularities are not isolated. The fibers of the Springer resolution of $\mathcal{N}$ have been studied by G. Pagnon (see e.g. [17]).

As in the Theorem 6.2 above, there should be a notion of G-versal deformation. In that case, if X is a non-singular slice of $G.x$ at x, where x is an irregular nilpotent element, the map γ should define a G-versal deformation of $X, x \to (\mathfrak{H}/W, 0)$.

More surprisingly in [16] simple elliptic singularities of surfaces of type $\tilde{D}_5$ (see definition in [19]) are obtained by intersecting the nilpotent variety of the Lie algebra $\mathcal{G} = sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ with a 4-dimensional special linear submanifold of $\mathcal{G}$. It is not known which type of singularities can appear by making simple constructions of this type.

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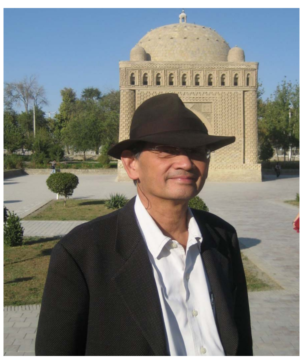

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